Two-Factor ANOVA

I. Introduction to Two Factor ANOVA

A. Factorial analysis of variance

B. No good general nonparametric tests

C. Experimental set-up:

- factor A, w/ a levels
- factor B, w/ b levels
- n = number of replicates for each combination of factors
  equal replication = <u>balanced</u> experimental design
- total # measurements = abn

D. Hypotheses:

H_0: No effect of factor A on response variable
H_A: Factor A does have an effect on response variable

H_0: No effect of factor B on response variable
H_A: Factor B does have an effect on response variable

H_0: No effect of interaction between factors A & B on response variable
H_A: Interaction between factors A & B does have an effect on response variable

E. Notation

1. individual measurements: X_{ijl} = l^{th} measurement in j^{th} level of B & i^{th} level of A
   1^{st} subscript = factor A 2^{nd} subscript = factor B 3^{rd} subscript = replicate

2. means: 4 kinds of means:
   cell means (cell ij): $\overline{X}_{ij}$
   level means of factor A: $\overline{X}_{i\cdot}$, e.g., $\overline{X}_{2\cdot}$
   level means of factor B: $\overline{X}_{\cdot j}$, e.g., $\overline{X}_{\cdot 1}$
   grand mean: $\overline{X}$

3. Summation: $\sum_i X_i$ $\sum_i \sum_j \sum_l X_{ijl} = \sum_{i=1}^{a}\left[\sum_{j=1}^{b}\left(\sum_{l=1}^{n} X_{ijl}\right)\right]$

II. Linear Model for Two-Factor ANOVA

$X_{ijl} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijl}$

where: $\varepsilon_{ijl} \sim N(0,\sigma^2)$ and independent of each other
H_0: $\alpha_i = 0$
H_0: $\beta_i = 0$
H_0: $(\alpha\beta)_{ij} = 0, \quad \forall ij$

III. Calculations:

A. Sums of Squares & Degrees of Freedom

<u>total</u>: total SS $= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{l=1}^{n}(X_{ijl} - \overline{X})^2$ total DF = $N - 1$

<u>cells</u>: cells SS $= \sum_{i=1}^{a}\sum_{j=1}^{b} n(\overline{X}_{ij} - \overline{X})^2$ cells DF = $ab - 1$

(continued)

III. **Calculations (continued):**

within-cells (error): error SS $= \sum_{i=1}^{a}\sum_{j=1}^{b}\left[\sum_{l=1}^{n}(X_{ijl} - \overline{X}_{ij})^2\right]$

error DF = $ab(n-1)$ = total DF − cells DF

factor A: factor A SS $= bn\sum_{i=1}^{a}(\overline{X}_{i\cdot} - \overline{X})^2$ factor A DF = $a - 1$

factor B: factor B SS $= an\sum_{j=1}^{b}(\overline{X}_{\cdot j} - \overline{X})^2$ factor B DF = $b - 1$

interaction between A & B:

$A \times B$ interaction SS = cells SS − factor A SS − factor B SS

$A \times B$ interaction DF = cells DF − factor A DF − factor B DF

or: $A \times B$ interaction DF = (factor A DF)(factor B DF) = $(a-1)(b-1)$

B. Mean Square = SS / DF

C. Machine formulas: see Zar, table 12.2

IV. **Testing Hypotheses & Models of Two-Factor ANOVA**

A. Model I ANOVA: both factors fixed

- test each of 3 null hypotheses (A, B, interaction) independently:
- significant interaction => difference among levels of a factor not constant w/ other factor
  => do not interpret factor effect if ∃ significant interaction effect
- Hypothesis testing w/ both factors fixed:
  all tests against within-cells MS (or error MS)
  - Factor A: $\dfrac{\text{factor } A \text{ MS}}{\text{error MS}} \sim F_{(a-1),ab(n-1)}$
  - Factor B: $\dfrac{\text{factor } B \text{ MS}}{\text{error MS}} \sim F_{(b-1),ab(n-1)}$
  - Interaction: $\dfrac{A \times B \text{ MS}}{\text{error MS}} \sim F_{(a-1)(b-1),ab(n-1)}$

B. Model II ANOVA: both factors random:

- test Factor A & B against interaction MS,
- test interaction against within-cells MS (error MS)
  - Factor A: $\dfrac{\text{factor } A \text{ MS}}{A \times B \text{ MS}} \sim F_{(a-1),(a-1)(b-1)}$
  - Factor B: $\dfrac{\text{factor } B \text{ MS}}{A \times B \text{ MS}} \sim F_{(b-1),(a-1)(b-1)}$
  - Interaction: $\dfrac{A \times B \text{ MS}}{\text{error MS}} \sim F_{(a-1)(b-1),ab(n-1)}$

C. Model III ANOVA (or mixed model ANOVA): one factor fixed, other random
→ e.g., for A fixed, B random:
• test fixed factor (A) against interaction MS,
• test random factor (B) and interaction against within-cells MS (error MS)

- Factor A: $\dfrac{\text{factor } A \text{ MS}}{A \times B \text{ MS}} \sim F_{(a-1),(a-1)(b-1)}$
- Factor B: $\dfrac{\text{factor } B \text{ MS}}{\text{error MS}} \sim F_{(b-1),ab(n-1)}$
- Interaction: $\dfrac{A \times B \text{ MS}}{\text{error MS}} \sim F_{(a-1)(b-1),ab(n-1)}$

V. **Multiple Comparisons Testing**

A. If two factor ANOVA => significant differences among levels of factor,
use multiple comparison test, Zar chapter 11 (e.g., Tukey test)
s^2 = within-cells MS
ν = within-cells DF
n = total # data per level, e.g., for factor A, n = bn

B. If significant interaction between factors,
DO NOT compare means of levels of a factor
instead, do multiple comparison test among <u>cell</u> <u>means</u>

VI. **Confidence Limits of Means**

same procedure as for single-factor ANOVA